■ RESEARCH ARTICLE

The Prime Factorization of the Product of Consecutive Factorials

Jimin Han[1], Hyeonjong Jeong, Juan Lim

QSI International School of Shenzhen, 8 Tai Zi Road, Shekou, Shenzhen, Guangdong, 518067, China, jmhan9999@gmail.com

ABSTRACT: In this paper, we have generalized the numerous conversion steps into one equation to find the exponents of each prime factor in the prime factorization of $\prod_{k=m}^{n} k!$, where m,n are any positive integers and $m<n$. We have also expressed $\prod_{k=m}^{n} k!$ using series notation with the generalized equation to demonstrate possible applications of the equation.

KEYWORDS: Mathematics; Algebra; Number Theory; Product Function; Prime Factorization; Generalization.

■ Introduction

A factorial (!) of a number signifies the product of 1 and every consecutive positive integer until the specified number.

$$N! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot N$$

The prime factorization of a factorial of the number N can be expressed as follows, where p_n indicates the prime factors of N and α_n indicates the corresponding exponent of each factor:

$$N! = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \cdot p_4^{\alpha_4} \cdot \cdots$$

To find the values of α_n, one can refer to Legendre's formula, also called de Polignac's formula, a commonly used equation for the prime factorization of a factorial.[1,4]

Doing so requires the use of the floor function. A floor function (⌊⌋) of a value denotes the largest integer smaller than or equal to the value in the floor function and is commonly used to find the quotient while excluding the remainder.

Now, when solving for α_n, multiples of the prime numbers that are less than or equal to N must be considered. For instance, to find the value of α_1, one would have to consider the multiples of p_1, the multiples of p_1^2 , the multiples of , and so on. If multiples of p_1 are to be counted once, then multiples of p_1^2 should be counted once more, for the multiples of p_1^2 would have two factors of p_1.

So, to count the multiples of p_1, one could use the floor function in the following manner:

Number of multiples of p_1 between 1 and N:

$$\left\lfloor \frac{N}{p_1} \right\rfloor$$

Then, to find the multiples of p_1^2 ,one could use the following:

$$\left\lfloor \frac{N}{p_1^2} \right\rfloor$$

Similarly, one could use the following to find the multiples of p_1^3

$$\left\lfloor \frac{N}{p_1^3} \right\rfloor$$

Continue this process until 0 comes out.

Adding the expressions yields the Legendre's formula:

$$\alpha_1 = \left\lfloor \frac{N}{p_1} \right\rfloor + \left\lfloor \frac{N}{p_1^2} \right\rfloor + \left\lfloor \frac{N}{p_1^3} \right\rfloor + \cdots$$

For instance, applying the Legendre's Formula to 10! indicates that

$$\alpha_1 = \left\lfloor \frac{10}{2} \right\rfloor + \left\lfloor \frac{10}{4} \right\rfloor + \left\lfloor \frac{10}{8} \right\rfloor = 5 + 2 + 1 = 8$$

We can verify this since $10!=1\cdot2\cdot3\cdot4\cdot\cdots\cdot10=2^8\cdot3^4\cdot5^2\cdot7$.

The generalized Legendre's formula is as follows:

$$\nu_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor,$$

where ν_p, the p-adic valuation[2], represents the exponent of the largest power of the prime number p that divides $n!$.

In this paper, using the process of Legendre's formula, a generalized form is suggested to find the exponent of a certain prime factor in multiple consecutive factorials.[1-4] In fact, from the superfactorial, which is explained in Section 8 "On the factors of superfactorials" in "The Generalized Superfactorial, Hyperfactorial, and Primorial functions"[3]

$$\mathrm{sf}(n) = 1! \cdot 2! \cdot 3! \cdot 4! \cdot \cdots \cdot n!,$$

the following expression is derived:

$$\nu_p(\mathrm{sf}(n)) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor \left(n + 1 - \frac{p^k}{2} \left(\left\lfloor \frac{n}{p^k} \right\rfloor + 1 \right) \right)$$

Furthermore, an advanced generalized form is proposed for finding the exponent of said prime factor in multiple consecutive integers, which may not necessarily progress from 1 to N.

DOI: 10.36838/v6i5.12

Methods

In the form of

$$\prod_{k=1}^{n} k! = 1! \cdot 2! \cdot 3! \cdot 4! \cdot \cdots \cdot n! = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \cdot p_4^{\alpha_4} \cdot \cdots,$$

the exponent of a prime factor of the prime factorization, $\alpha_1, \alpha_2, \alpha_3 \ldots$, can be found by using the following equation. Note that α_i and b_i are integers used in the calculations, and i and k represent indices.

$$\alpha_i = (n+1)(a_1 + a_2 + \cdots) - \left(p_i^1 \cdot \frac{a_1(a_1+1)}{2} + p_i^2 \cdot \frac{a_2(a_2+1)}{2} + \cdots\right)$$

$$= \sum_{k=1}^{\infty} a_k \left((n+1) - \frac{(a_k+1)p_i^k}{2}\right),$$

where $a_1 = \left\lfloor \frac{n}{p} \right\rfloor$, and $a_k = \left\lfloor \frac{a_{k-1}}{p} \right\rfloor$,

or $a_k = \left\lfloor \frac{n}{p^k} \right\rfloor$

in explicit form.

This formula is used to find the value of each α value.

For example, let the value of n be 1000:

$$1! \cdot 2! \cdot 3! \cdot 4! \cdot \ldots \cdot 999! \cdot 1000! = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot \cdots$$

To find the exponent of 2, start from α_1:

$$a_1 = \left\lfloor \frac{1000}{2} \right\rfloor = 500$$

$$a_2 = \left\lfloor \frac{500}{2} \right\rfloor = 250$$

$$a_3 = \left\lfloor \frac{250}{2} \right\rfloor = 125$$

$$a_4 = \left\lfloor \frac{125}{2} \right\rfloor = 62$$

$$a_5 = \left\lfloor \frac{62}{2} \right\rfloor = 31$$

$$a_6 = \left\lfloor \frac{31}{2} \right\rfloor = 15$$

$$a_7 = \left\lfloor \frac{15}{2} \right\rfloor = 7$$

$$a_8 = \left\lfloor \frac{7}{2} \right\rfloor = 3$$

$$a_9 = \left\lfloor \frac{3}{2} \right\rfloor = 1$$

$$a_{10} = \left\lfloor \frac{1}{2} \right\rfloor = 0$$

$$\vdots$$

$$a_\infty = \left\lfloor \frac{0}{2} \right\rfloor = 0$$

$$\alpha_1 = (1000+1)(500+250+125+62+31+15+7+3+1)$$
$$-\left(\frac{2 \cdot 500 \cdot 501}{2} + \frac{2^2 \cdot 250 \cdot 251}{2} + \frac{2^3 \cdot 125 \cdot 126}{2} + \frac{2^4 \cdot 62 \cdot 63}{2} + \frac{2^5 \cdot 31 \cdot 32}{2} + \frac{2^6 \cdot 15 \cdot 16}{2} + \frac{2^7 \cdot 7 \cdot 8}{2} + \frac{2^8 \cdot 3 \cdot 4}{2} + \frac{2^9 \cdot 1 \cdot 2}{2}\right) = 495562$$

Similarly, the steps to find the exponent of 3 are as follows:

$$a_1 = \left\lfloor \frac{1000}{3} \right\rfloor = 333$$

$$a_2 = \left\lfloor \frac{333}{3} \right\rfloor = 111$$

$$a_3 = \left\lfloor \frac{111}{3} \right\rfloor = 37$$

$$a_4 = \left\lfloor \frac{37}{3} \right\rfloor = 12$$

$$a_5 = \left\lfloor \frac{12}{3} \right\rfloor = 4$$

$$a_6 = \left\lfloor \frac{4}{3} \right\rfloor = 1$$

$$a_7 = \left\lfloor \frac{1}{3} \right\rfloor = 0$$

$$\vdots$$

$$a_\infty = \left\lfloor \frac{0}{3} \right\rfloor = 0$$

$$\alpha_2 = (1000+1)(333+111+37+12+4+1)$$
$$-\left(\frac{3 \cdot 333 \cdot 334}{2} + \frac{3^2 \cdot 111 \cdot 112}{2} + \frac{3^3 \cdot 37 \cdot 38}{2} + \frac{3^4 \cdot 12 \cdot 13}{2} + \frac{3^5 \cdot 4 \cdot 5}{2} + \frac{3^6 \cdot 1 \cdot 2}{2}\right)$$
$$= 247263$$

$$\therefore\ 1! \cdot 2! \cdot 3! \cdot 4! \cdot \cdots \cdot 999! \cdot 1000! = 2^{495562} \cdot 3^{247263} \cdot \ldots$$

The exponent of the prime numbers can still be found even if the list of factorials does not start from 1.

This can be represented in the form of

$$\prod_{k=m}^{n} k! = m! \cdot (m+1)! \cdot (m+2)! \cdot (m+3)! \cdot \cdots \cdot n! = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \cdot p_4^{\alpha_4} \cdots$$

For this case, the exponent of a prime factor of the prime factorization, $\alpha_1, \alpha_2, \alpha_3 \ldots$, can be found by using the following equation.

$$\alpha_n = \sum_{i=1}^{\infty} \left(a_i(n+1) - b_i m - \frac{(a_i(a_i+1) - b_i(b_i+1))}{2} p_k^i\right),$$

where $a_1 = \left\lfloor \frac{n}{p} \right\rfloor$ and $a_i = \left\lfloor \frac{a_{i-1}}{p} \right\rfloor$, and $b_1 = \left\lfloor \frac{m}{p} \right\rfloor$,.

For example, let the value of m be 251 and n be 667.

$$251! \cdot 252! \cdot 253! \ \ldots\ 666! \cdot 667!$$

To find the exponent of 2, begin by finding α_1:

$$a_1 = \left\lfloor \frac{251}{2} \right\rfloor = 125$$

$$a_2 = \left\lfloor \frac{125}{2} \right\rfloor = 62$$

$$a_3 = \left\lfloor \frac{62}{2} \right\rfloor = 31$$

$$a_4 = \left\lfloor \frac{31}{2} \right\rfloor = 15$$

$$a_5 = \left\lfloor \frac{15}{2} \right\rfloor = 7$$

$$a_6 = \left\lfloor \frac{7}{2} \right\rfloor = 3$$

$$a_7 = \left\lfloor \frac{3}{2} \right\rfloor = 1$$

$$a_8 = \left\lfloor \frac{1}{2} \right\rfloor = 0$$

$$\vdots$$

$$a_\infty = \left\lfloor \frac{0}{2} \right\rfloor = 0$$

Then, repeat the above process for b_i, starting from i=1:

$$b_1 = \left\lfloor \frac{667}{2} \right\rfloor = 333$$

$$b_2 = \left\lfloor \frac{333}{2} \right\rfloor = 166$$

$$b_3 = \left\lfloor \frac{166}{2} \right\rfloor = 83$$

$$b_4 = \left\lfloor \frac{83}{2} \right\rfloor = 41$$

$$b_5 = \left\lfloor \frac{41}{2} \right\rfloor = 20$$

$$b_6 = \left\lfloor \frac{20}{2} \right\rfloor = 10$$

$$b_7 = \left\lfloor \frac{10}{2} \right\rfloor = 5$$

$$b_8 = \left\lfloor \frac{5}{2} \right\rfloor = 2$$

$$b_9 = \left\lfloor \frac{2}{2} \right\rfloor = 1$$

$$b_{10} = \left\lfloor \frac{1}{2} \right\rfloor = 0$$

$$\vdots$$

$$b_\infty = \left\lfloor \frac{0}{2} \right\rfloor = 0$$

$$\begin{aligned}
\alpha_1 = {} & 333(667+1) - 125 \cdot 251 - \frac{\left(333(333+1) - 125(125+1)\right)}{2} \cdot 2 \\
& +166(667+1) - 62 \cdot 251 - \frac{\left(166(166+1) - 62(62+1)\right)}{2} \cdot 4 \\
& +83(667+1) - 31 \cdot 251 - \frac{\left(83(83+1) - 31(31+1)\right)}{2} \cdot 8 \\
& +41(667+1) - 15 \cdot 251 - \frac{\left(41(41+1) - 15(15+1)\right)}{2} \cdot 16 \\
& +20(667+1) - 7 \cdot 251 - \frac{\left(20(20+1) - 7(7+1)\right)}{2} \cdot 32 \\
& +10(667+1) - 3 \cdot 251 - \frac{\left(10(10+1) - 3(3+1)\right)}{2} \cdot 64 \\
& +5(667+1) - 1 \cdot 251 - \frac{\left(5(5+1) - 1(1+1)\right)}{2} \cdot 128 \\
& +2(667+1) - 0 \cdot 251 - \frac{\left(2(2+1) - 0(0+1)\right)}{2} \cdot 256 \\
& +1(667+1) - 0 \cdot 251 - \frac{\left(1(1+1) - 0(0+1)\right)}{2} \cdot 512 \\
= {} & 189392.
\end{aligned}$$

Similarly, the steps to find the exponent of 3 are as follows:

$$a_1 = \left\lfloor \frac{667}{3} \right\rfloor = 222$$

$$a_2 = \left\lfloor \frac{222}{3} \right\rfloor = 74$$

$$a_3 = \left\lfloor \frac{74}{3} \right\rfloor = 24$$

$$a_4 = \left\lfloor \frac{24}{3} \right\rfloor = 8$$

$$a_5 = \left\lfloor \frac{8}{3} \right\rfloor = 2$$

$$a_6 = \left\lfloor \frac{2}{3} \right\rfloor = 0$$

$$\vdots$$

$$a_\infty = \left\lfloor \frac{0}{3} \right\rfloor = 0$$

Then, repeat the above process for b_i, starting from i=1:

$$b_1 = \left\lfloor \frac{251}{3} \right\rfloor = 83$$

$$b_2 = \left\lfloor \frac{83}{3} \right\rfloor = 27$$

$$b_3 = \left\lfloor \frac{27}{3} \right\rfloor = 9$$

$$b_4 = \left\lfloor \frac{9}{3} \right\rfloor = 3$$

$$b_5 = \left\lfloor \frac{3}{3} \right\rfloor = 1$$

$$b_6 = \left\lfloor \frac{1}{3} \right\rfloor = 0$$

$$\vdots$$

$$b_\infty = \left\lfloor \frac{0}{3} \right\rfloor = 0$$

$$\begin{aligned}
\alpha_2 = {} & 222(667+1) - 83 \cdot 251 - \frac{\left(222(222+1) - 83(83+1)\right)}{2} \cdot 3 \\
& +74(667+1) - 27 \cdot 251 - \frac{\left(74(74+1) - 27(27+1)\right)}{2} \cdot 9 \\
& +24(667+1) - 9 \cdot 251 - \frac{\left(24(24+1) - 9(9+1)\right)}{2} \cdot 27 \\
& +8(667+1) - 3 \cdot 251 - \frac{\left(8(8+1) - 3(3+1)\right)}{2} \cdot 81 \\
& +2(667+1) - 1 \cdot 251 - \frac{\left(2(2+1) - 1(1+1)\right)}{2} \cdot 243 = 94392.
\end{aligned}$$

$$\therefore\ 251! \cdot 252! \cdot 253! \cdot \cdots \cdot 666! \cdot 667! = 2^{189392} \cdot 3^{94392} \cdot \ldots$$

Results and Discussion

The following is a series of steps to derive the presented formula.

First, set up the equation: $1! \cdot 2! \cdot 3! \cdot 4! \cdots n! = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \cdot p_4^{\alpha_4} \cdots p_k^{\alpha_k}$, where p_k is a prime factor and α_k is the exponent of the prime factor.

$$\begin{gathered}
1 \ \cdot \\
1 \cdot 2 \ \cdot \\
1 \cdot 2 \cdot 3 \ \cdot \\
1 \cdot 2 \cdot 3 \cdot 4 \ \cdot \\
\vdots \\
1 \cdot 2 \cdot 3 \cdot 4 \cdots n
\end{gathered}$$

Notice that 1 appears n times, 2 appears n-1 times, and so on. Thus, the product of the above can be written as the following:

$$1^n \cdot 2^{n-1} \cdot 3^{n-2} \cdot 4^{n-3} \cdot \cdots \cdot n^1 \quad \cdots (1)$$

Then, divide the work into several cases in each base number p based on the exponents of p since numbers among multiples of p may contain two or more p as factors.

In the form of (1), choose and arrange the numbers with at least one p as a factor:

$$(p_k)^{(n+1)-p_k} \quad (2p_k)^{(n+1)-2p_k} \quad (3p_k)^{(n+1)-3p_k} \quad \cdots \quad \left(\left\lfloor \frac{n}{p_k} \right\rfloor p_k\right)^{(n+1)-\left\lfloor \frac{n}{p_k} \right\rfloor p_k}$$

Do the same for the numbers with at least two p_k as factors:

$$({p_k}^2)^{(n+1)-{p_k}^2} \quad (2{p_k}^2)^{(n+1)-2{p_k}^2} \quad (3{p_k}^2)^{(n+1)-3{p_k}^2} \cdots \quad \left(\left\lfloor \frac{n}{p_k^2} \right\rfloor {p_k}^2\right)^{(n+1)-\left\lfloor \frac{n}{p_k^2} \right\rfloor {p_k}^2}$$

Repeat the above process for different values of $p_k^{a_k}$

DOI: 10.36838/v6i5.12

until $\left\lfloor \frac{n}{p_k^{a_k}} \right\rfloor$ reaches zero (that is, when the denominator exceeds the numerator).

From the prior steps, the base value of the last term, $\left\lfloor \frac{n}{p_k} \right\rfloor$, is utilized to calculate the total number of multiples of a prime number p between 1 and n. For identifying the largest multiple of p that is less than n, the floor function $\left\lfloor \frac{n}{p_k} \right\rfloor$ is applied, and the result is multiplied by p. Doing so ensures that only the multiples of p less than n are considered.

Furthermore, the total power of p is found considering the above list of expressions. This is achieved by systematically multiplying all of the values, which add up the exponents because of the property of exponents.

Then, we apply the formula for the sum of an arithmetic series to represent the addition of the consecutive exponents. This process simplifies the calculation into following.

$$\frac{\left\lfloor \frac{n}{p_k} \right\rfloor \cdot \left(2(n+1) - p_k - p_k\left\lfloor \frac{n}{p_k} \right\rfloor\right) + \left\lfloor \frac{n}{p_k^2} \right\rfloor\left(2(n+1) - p_k^2 - p_k^2\left\lfloor \frac{n}{p_k^2} \right\rfloor\right) \cdots}{2}$$

After reorganizing the expression, we can get this:

$$\frac{2(n+1)\left(\left\lfloor \frac{n}{p_k} \right\rfloor + \left\lfloor \frac{n}{p_k^2} \right\rfloor \cdots\right) - \left(p_k\left\lfloor \frac{n}{p_k} \right\rfloor + p_k^2\left\lfloor \frac{n}{p_k^2} \right\rfloor \cdots\right) - \left(p_k\left\lfloor \frac{n}{p_k} \right\rfloor^2 + p_k^2\left\lfloor \frac{n}{p_k^2} \right\rfloor^2 \cdots\right)}{2}$$

To simplify, let $a_1 = \left\lfloor \frac{n}{p_k} \right\rfloor$ and $a_n = \left\lfloor \frac{a_{n-1}}{p_k} \right\rfloor$ since $\frac{a}{b^n} = \left\lfloor \frac{a}{b^n} \right\rfloor + \left\{ \frac{a}{b^n} \right\}$ can be reorganized as $\frac{a}{b^n} - \left\{ \frac{a}{b^n} \right\} = \left\lfloor \frac{a}{b^n} \right\rfloor$ and

$$\left\lfloor \frac{\left\lfloor \frac{a}{b^n} \right\rfloor}{b} \right\rfloor = \left\lfloor \frac{\frac{a}{b^n} - \left\{ \frac{a}{b^n} \right\}}{b} \right\rfloor = \left\lfloor \frac{a}{b^{n+1}} - \frac{\left\{ \frac{a}{b^n} \right\}}{b} \right\rfloor = \left\lfloor \frac{a}{b^{n+1}} \right\rfloor,$$

Then,

$$(n+1)(a_1 + a_2 \cdots) - \frac{1}{2}(p_k a_1(a_1+1) + p_k^2 a_2(a_2+1) \cdots)$$

At the end, using notations, the above can be written as the following:

$$\sum_{i=1}^{\infty}(a_i(n+1) - \frac{a_i(a_i+1)p_k^i}{2})$$

Furthermore, we can derive a formula for the product of any consecutive factorials (that is, one that does not necessarily start from 1!).

Let's say that the list of the factorials is $m!\cdot(m+1)!\cdot(m+2)!\cdot \cdots \cdot(n-1)!\cdot n!$. The product of the consecutive 1! through n! should be divided by the product of the consecutive 1! through $(m-1)$! to find the specific product of consecutive factorials from m! to n!. For instance, the product of all consecutive factorials from 4! to 6! can be expressed in the following manner: 4!·5!·6!=(1!·2!·3!·4!·5!·6!)÷(1!·2!·3!).

In fact, the formula for this can be immediately derived using the following:

$$v_p(m! \cdot \cdots \cdot n!) = v_p\left(\frac{m!\,\mathrm{sf}(n)}{\mathrm{sf}(m)}\right) = v_p(m!) + v_p(\mathrm{sf}(n)) - v_p(\mathrm{sf}(m))$$

Nonetheless, the formal derivation steps are as follows:

First, the term for a particular prime number p_k from 1! to n! can be expressed as $p_k^{n+1-p_k}$. Likewise, the term for a particular prime number p from 1! to $(m-1)$! can be expressed as $p_k^{(m-1)+1-p_k}$. To find the total exponents from m! to n!, subtract the exponent of $p_k^{(m-1)+1-p_k}$ from $p_k^{n+1-p_k}$: $(n+1-p_k) - ((m-1)+1-p_k) = n-m+1$. Here, notice that there exist $\left\lfloor \frac{m}{p_k} \right\rfloor$ terms for the exponents, as explained in the derivation to the previous formula; multiply them together to get $\left\lfloor \frac{m}{p_k} \right\rfloor \cdot (n-m+1)$.

Also, we know that from m! to n!, there are $\left\lfloor \frac{n}{p_k} \right\rfloor - \left\lfloor \frac{m}{p_k} \right\rfloor$ terms.

Plus, for $\left\lfloor \frac{m}{p_k} \right\rfloor + 1$ (the term after m!), we have the exponent $(n+1) - p_k\left(\left\lfloor \frac{m}{p_k} \right\rfloor + 1\right)$, and for $\left\lfloor \frac{n}{p_k} \right\rfloor$, we have the exponent, $(n+1) - p_k\left(\left\lfloor \frac{n}{p_k} \right\rfloor\right)$ as stated in the derivation above.

Also, putting these into the arithmetic sum formula yields

$$\frac{\left(2(n+1) - p_k\left(\left\lfloor \frac{m}{p_k} \right\rfloor + 1\right) - p_k\left\lfloor \frac{n}{p_k} \right\rfloor\right) \cdot \left(\left\lfloor \frac{n}{p_k} \right\rfloor - \left\lfloor \frac{m}{p_k} \right\rfloor\right)}{2}.$$

Now, adding the two together, the following is derived:

$$\left\lfloor \frac{m}{p_k} \right\rfloor \cdot (n-m+1) + \frac{\left(2(n+1) - p_k\left(\left\lfloor \frac{m}{p_k} \right\rfloor + 1\right) - p_k\left\lfloor \frac{n}{p_k} \right\rfloor\right) \cdot \left(\left\lfloor \frac{n}{p_k} \right\rfloor - \left\lfloor \frac{m}{p_k} \right\rfloor\right)}{2}$$

Since i th power of p_k can be inserted in the denominator until $\left\lfloor \frac{m}{p_k^i} \right\rfloor$ reaches zero, the above expression can be expressed using sigma:

$$\sum_{i=1}^{\infty}\left[\left\lfloor \frac{m}{p_k^i} \right\rfloor \cdot (n-m+1) + \frac{\left(2(n+1) - p_k^i\left(\left\lfloor \frac{m}{p_k^i} \right\rfloor + 1\right) - p_k^i\left\lfloor \frac{n}{p_k^i} \right\rfloor\right) \cdot \left(\left\lfloor \frac{n}{p_k^i} \right\rfloor - \left\lfloor \frac{m}{p_k^i} \right\rfloor\right)}{2}\right]$$

Reorganization:

$$\sum_{i=1}^{\infty}\left[\left\lfloor \frac{m}{p_k^i} \right\rfloor \cdot (n-m+1) + \frac{2(n+1)\left(\left\lfloor \frac{n}{p_k^i} \right\rfloor - \left\lfloor \frac{m}{p_k^i} \right\rfloor\right) - p_k^i\left(\left\lfloor \frac{n}{p_k^i} \right\rfloor^2 - \left\lfloor \frac{m}{p_k^i} \right\rfloor^2 + \left\lfloor \frac{n}{p_k^i} \right\rfloor - \left\lfloor \frac{m}{p_k^i} \right\rfloor\right)}{2}\right]$$

This is the most we can simplify the expression:

$$\sum_{i=1}^{\infty}\left[(n+1)\left\lfloor \frac{n}{p_k^i} \right\rfloor - m\left\lfloor \frac{m}{p_k^i} \right\rfloor - \frac{\left\lfloor \frac{n}{p_k^i} \right\rfloor\left(\left\lfloor \frac{n}{p_k^i} \right\rfloor + 1\right) - \left\lfloor \frac{m}{p_k^i} \right\rfloor\left(\left\lfloor \frac{m}{p_k^i} \right\rfloor + 1\right)}{2}p_k^i\right]$$

Let $a_1 = \left\lfloor \frac{n}{p_k} \right\rfloor$ and $a_n = \left\lfloor \frac{a_{n-1}}{p_k} \right\rfloor$, and let $b_1 = \left\lfloor \frac{m}{p_k} \right\rfloor$ and $b_n = \left\lfloor \frac{b_{n-1}}{p_k} \right\rfloor$.

Hence, the final formula is the following:

$$\sum_{i=1}^{\infty}\left[a_i(n+1) - b_i m - \frac{(a_i(a_i+1) - b_i(b_i+1))}{2}p_k^i\right]$$

Also, this mathematical expression can be simply driven using the first generalized form.

Let $1! \cdot 2! \cdot 3! \cdot 4! \cdots n! = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \cdot p_4^{\alpha_4} \cdots p_k^{\alpha_k}$

and $1! \cdot 2! \cdot 3! \cdot 4! \cdots (m-1)! = p_1^{\beta_1} \cdot p_2^{\beta_2} \cdot p_3^{\beta_3} \cdot p_4^{\beta_4} \cdots p_k^{\beta_k}$.

Then,

$$m! \cdot (m+1)! \cdot (m+2)! \cdot \cdots \cdot (n-1)! \cdot n! = \frac{1! \cdot 2! \cdot 3! \cdot 4! \cdots n!}{1! \cdot 2! \cdot 3! \cdot 4! \cdots (m-1)!} = p_1^{\alpha_1-\beta_1} \cdot p_1^{\alpha_2-\beta_2} \cdot p_1^{\alpha_3-\beta_3} \cdot p_1^{\alpha_4-\beta_4} \cdots p_1^{\alpha_k-\beta_k}$$

Also, as already derived above, the formula for the exponents of $1! \cdot 2! \cdot 3! \cdot 4! \cdots n!$ is $\sum_{i=1}^{\infty}\left(a_i(n+1) - \frac{a_i(a_i+1)p_k^i}{2}\right)$. Likewise, the formula for the exponents of $1! \cdot 2! \cdot 3! \cdot 4! \cdots (m-1)!$ is $\sum_{i=1}^{\infty}\left(b_i m - \frac{b_i(b_i+1)p_k^i}{2}\right)$.

Since they are exponents, subtracting the two together will finally yield the formula:

$$\sum_{i=1}^{\infty}\left(\left(a_i(n+1) - \frac{a_i(a_i+1)p_k^i}{2}\right) - \left(b_i m - \frac{b_i(b_i+1)p_k^i}{2}\right)\right)$$
$$= \sum_{i=1}^{\infty}\left(a_i(n+1) - b_i m - \frac{\left(a_i(a_i+1) - b_i(b_i+1)\right)}{2} p_k^i\right)$$

Conclusion

In this paper, a series of steps for deriving prime factorization from a product of consecutive factorials, $\prod_{k=1}^{n} k! = 1! \cdot 2! \cdot 3! \cdot 4! \cdot \cdots \cdot n!$, is simplified into one simple equation as follows: $\sum_{i=1}^{\infty}\left(a_i(n+1) - \frac{a_i(a_i+1)p_k^i}{2}\right)$, where p_k represents the kth prime number, and α_i represents $\left\lfloor \frac{a_{i-1}}{p_k} \right\rfloor$ and $a_1 = \left\lfloor \frac{n}{p_k} \right\rfloor$. .This paper also derived a formula for the exponents of prime factors of prime factorization of any random consecutive factorials, $\prod_{k=m}^{n} k! = m! \cdot (m+1)! \cdot (m+2)! \cdot (m+3)! \cdot \cdots \cdot n!$,:

$\prod_{k=m}^{n} k! = m! \cdot (m+1)! \cdot (m+2)! \cdot (m+3)! \cdot \cdots \cdot n!$,where α_i represents $\left\lfloor \frac{a_{i-1}}{p_k} \right\rfloor$ and $a_1 = \left\lfloor \frac{n}{p_k} \right\rfloor$ and b_i represents $\left\lfloor \frac{b_{i-1}}{p_k} \right\rfloor$ and $b_1 = \left\lfloor \frac{m}{p_k} \right\rfloor$.

Through this study, our paper provides a generalized formula for deriving the prime factorization of multiple consecutive factorials. Nonetheless, there is a clear extent to which this study is valid: this paper only regards one specific type of factorial. Hence, a formula for the prime factorization of a multiple of double factorials could be derived through further research.

Acknowledgments

We would like to thank our parents for providing such a great opportunity and environment that enabled us to conduct this research across international borders.

We would also like to thank Songtao Mao, a Ph.D. student, for providing valuable feedback that helped improve the quality of our paper.

References

1. Dickson, L. E. (1919). *History of the Theory of Numbers*. https://doi.org/10.5962/t.174869
2. Eremin, G. (2019). Legendre's formula and $ p $-adic analysis. arXiv preprint https://arxiv.org/abs/1907.11902
3. Raman, Vignesh. (2020). The Generalized Superfactorial, Hyperfactorial, and Primorial functions. https://doi.org/10.48550/arXiv.2012.00882
4. Zhu, X., & Niu, C. (2023). The p-adic valuation of $C_n(l)=\prod_{k=1}^{n}\prod_{j=0}^{l}(2k+2j-1)$. SCIREA Journal of Mathematics, 8(1). https://doi.org/10.54647/mathematics110379

Authors

Jimin Han is a junior at QSI International School of Shenzhen. As an active student body member, he regularly participates in community service opportunities and math competitions. He plans to study engineering in college.

Hyeonjong (Henry) Jeong is a junior at Korea International School, Jeju Campus. Serving as an officer for the school's Mu Alpha Theta club, he participates in various school-led and independent math competitions. He aspires to pursue a career in biochemistry, particularly in the field of medicine.

Juan Lim is a sophomore at Korea International School Jeju. As a member of the robotics team, engineering club, and Mu Alpha Theta, he participates in various STEM projects and competitions. He plans to study engineering in college.

DOI: 10.36838/v6i5.12